Exact Casimir energies at nonzero temperature: Validity of proximity force approximation and interaction of semitransparent spheres

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In this paper, dedicated to the career of Tom Erber, we consider the Casimir interaction between weakly coupled bodies at nonzero temperature. For the case of semitransparent bodies, that is, ones described by δ-function potentials, we first examine the interaction between an infinite plane and an arbitrary curved surface. In weak coupling, such an interaction energy coincides with the exact form of the proximity force approximation obtained by summing the interaction between opposite surface elements at arbitrary temperature. This result generalizes a theorem proved recently by Decca et al. We also obtain exact closed-form results for the Casimir energy at arbitrary temperature for weakly-coupled semitransparent spheres.

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I. INTRODUCTION

Since the earliest calculations of fluctuation forces between bodies [1], that is, Casimir or quantum vacuum forces, multiple scattering methods have been employed. Rather belatedly, it has been realized that such methods could be used to obtain accurate numerical results in many cases [2–5]. These results allow us to transcend the limitations of the proximity force theorem (PFT) [6, 7], and so make better comparison with experiment, which typically involve curved surfaces. (For a review of the experimental situation, see Ref. [8].)

The multiple scattering formalism, which is in principle exact, dates back at least into the 1950s [9, 10]. Particularly noteworthy is the seminal work of Balian and Duplantier [11]. (For more complete references see Ref. [12].) This technique, which has been brought to a high state of perfection by Emig et al. [5], has concentrated on numerical results for the Casimir forces between conducting and dielectric bodies such as spheres and cylinders. For recent impressive numerical results for metals and dielectrics see Refs. [13, 14]. Our group has noticed that the multiple-scattering method can yield exact, closed-form results for bodies that are weakly coupled to the quantum field [12, 15]. (That is, we are carrying out first-order perturbation theory in the background potential. For early examples of this in the Casimir context, see Ref. [16].) This allows an exact assessment of the range of applicability of the PFT. The calculations there, however, as those in recent extensions of our methodology [17], have been restricted to scalar fields with δ-function potentials, so-called semitransparent bodies. (These are closely related to plasma shell models [3, 18–21].) The technique was recently extended to dielectric bodies [22, 23], characterized by a permittivity ε. Strong coupling would mean a perfect metal, $\varepsilon \to \infty$, while weak coupling means that ε is close to unity.

In this paper we will extend the weak-coupling formalism to the situation of nonzero temperature. This extension is extremely straightforward. We then apply the general formula to the case of an arbitrarily curved semitransparent surface above an infinite semitransparent plane. Remarkably, the result coincides with the use of the so-called proximity force approximation (PFA), which in its general form is exact in this case for all separations between the surfaces and for all temperatures. We also obtain exact closed-form results for the forces between separated spherical shells for all temperatures. In the Appendix we discuss exact formulas for arbitrary positive and negative powers of the distances between points on two spheres, needed for such calculations.

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II. MULTIPLE SCATTERING DERIVATION OF VACUUM ENERGY BETWEEN WEAKLY COUPLED POTENTIALS

The quantum vacuum energy for the interaction mediated by a massless scalar field between two nonoverlapping potentials $V_1(\mathbf{x})$ and $V_2(\mathbf{x})$ is

$$E = -\frac{i}{2}\mathrm{Tr}\,\ln(1 - V_1G_1V_2G_2), \tag{2.1}$$

in terms of the single potential Green's functions

$$G_i = (1 + G_0V_i)^{-1}G_0. \tag{2.2}$$

The free Green's function, satisfying

$$-\partial^2 G_0(x - x') = \delta(x - x'), \tag{2.3}$$

has the explicit form

$$G_0(x - x') = \int \frac{d\omega}{2\pi}\mathcal{G}_0(\mathbf{r} - \mathbf{r}', \omega)e^{-i\omega(t-t')}, \tag{2.4}$$

where the time-Fourier transform is

$$\mathcal{G}_0(\mathbf{r} - \mathbf{r}', i\zeta) = \frac{e^{-|\zeta||\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|}, \tag{2.5}$$

where we have performed the Euclidean rotation $\omega \to i\zeta$.

For weak potentials, the energy (2.1) simplifies dramatically:

$$E \approx \frac{i}{2}\mathrm{Tr}\,V_1G_0V_2G_0 = -\frac{1}{64\pi^3}\int (d\mathbf{r})(d\mathbf{r}')\frac{V_1(\mathbf{r})V_2(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{2.6}$$

At finite temperature the integral over imaginary frequency becomes the Matsubara sum:

$$\int_{-\infty}^{\infty} \frac{d\zeta}{2\pi}\frac{e^{-2|\zeta||\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|^2} \to T\sum_{m=-\infty}^{\infty}\frac{e^{-4\pi T|m||\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|^2} = \frac{T}{|\mathbf{r} - \mathbf{r}'|^2}\coth 2\pi T|\mathbf{r} - \mathbf{r}'|, \tag{2.7}$$

so the interaction energy becomes

$$E_T = -\frac{T}{32\pi^2}\int (d\mathbf{r})(d\mathbf{r}')V_1(\mathbf{r})V_2(\mathbf{r}')\frac{\coth 2\pi T|\mathbf{r} - \mathbf{r}'|}{|\mathbf{r} - \mathbf{r}'|^2}, \tag{2.8}$$

which evidently reduces to Eq. (2.6) for $T = 0$.

III. PARALLEL PLATES

For parallel, semitransparent plates, separated by a distance a, where the potentials are

$$V_1(\mathbf{r}) = \lambda_1\delta(z), \quad V_2(\mathbf{r}) = \lambda_2\delta(z - a), \tag{3.1}$$

the integrals in Eq. (2.6) are readily carried out, with the resulting energy per unit area A, $\mathcal{E} = E/A$:

$$\mathcal{E} = -\frac{\lambda_1\lambda_2}{32\pi^2 a}. \tag{3.2}$$

This well-known result holds even if *one* of the plates has a finite area A. At finite temperature the result is

$$\mathcal{E}_T = -\frac{\lambda_1\lambda_2 T}{16\pi}\int_{2\pi Ta}^{\infty}\frac{dx}{x}\coth x. \tag{3.3}$$

The energy is ambiguous because it depends on the arbitrarily chosen upper limit. However, it corresponds to a well-defined pressure between the plates,

$$P_T = -\frac{\partial}{\partial a}\mathcal{E}_T = -\frac{\lambda_1\lambda_2 T}{16\pi a}\coth 2\pi Ta. \tag{3.4}$$

IV. INTERACTION BETWEEN AN INFINITE PLANE AND AN ARBITRARILY CURVED SURFACE: PFA

Now consider the interaction between a semitransparent plane, described by the potential

$$V_1(\mathbf{r}) = \lambda_1 \delta(z), \tag{4.1}$$

and an arbitrary curved surface S, which does not intersect the plane $z = 0$, which corresponds to the potential

$$V_2(\mathbf{r}) = \lambda_2 \delta(z - s(x, y)), \tag{4.2}$$

where $z = s(x, y)$ is the equation of the surface. Then, from Eq. (2.8) it is immediate that the energy is (the upper limit of the x integration is again physically irrelevant)

$$E_T = -\frac{\lambda_1 \lambda_2 T}{16\pi} \int dS \int_{2\pi T z(S)} dx \frac{\coth x}{x}, \tag{4.3}$$

where the area integral is over the curved surface. This is precisely what one means by the proximity force approximation, where one sums energies between adjacent elements treated as parallel plates:

$$E_{\text{PFA}} = \int dS \mathcal{E}_{\parallel}(z(S)), \tag{4.4}$$

in view of Eq. (3.3). This is in fact just the theorem proved by Decca et al. [24], who were considering gravitational and Yukawa type forces, but we see it applies to any central force.

For example, the above, exact formula for weakly-coupled semitransparent surfaces says that the force on such a sphere, of radius a, the center of which is a distance Z above a semitransparent plane is

$$F_T = -\frac{\partial E_T}{\partial Z} = -\frac{\lambda_1 \lambda_2 a T}{8} \int_{2\pi T(Z-a)}^{2\pi T(Z+a)} \frac{du}{u} \coth u. \tag{4.5}$$

The zero-temperature limit of this is

$$F = -\frac{\lambda_1 \lambda_2}{8\pi} \frac{a^2}{Z^2 - a^2}, \tag{4.6}$$

which may be alternatively derived from the zero-temperature energy

$$E = -\frac{\lambda_1 \lambda_2 a^2}{16\pi} \int_{-1}^{1} \frac{d\cos\theta}{Z + a\cos\theta} = -\frac{\lambda_1 \lambda_2 a}{16\pi} \ln \frac{Z + a}{Z - a}, \tag{4.7}$$

again, the exact PFA result.

V. INTERACTION BETWEEN TWO SEMITRANSPARENT SPHERES AT NONZERO TEMPERATURE

Consider now two spheres, of radius a and b, respectively, with a distance between their centers $R > a + b$. In terms of local coordinates with origins at the centers of the two spheres, the semitransparent potentials are

$$V_1 = \lambda_1 \delta(r - a), \quad V_2 = \lambda_2 \delta(r' - b), \tag{5.1}$$

and let us further suppose that $\mathbf{R}$ lies along the z axis of both coordinate systems. Then the squared distance between points on the spheres is

$$|\mathbf{r} - \mathbf{r}'|^2 = R^2 + a^2 + b^2 - 2ab\cos\gamma - 2R(a\cos\theta - b\cos\theta'), \tag{5.2}$$

in terms of polar angles in the two spheres, where the cosine of the angle between the two radial vectors locating the points is

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'). \tag{5.3}$$

We insert this into the expression for the energy (2.8), obtaining

$$E = -\frac{\lambda_1\lambda_2 T}{32\pi^2}a^2b^2\int d\Omega\, d\Omega' \frac{\coth 2\pi T|\mathbf{r}-\mathbf{r}'|}{|\mathbf{r}-\mathbf{r}'|^2}. \tag{5.4}$$

It seems difficult to proceed in general, but we can work out a low temperature expansion using

$$\coth y = \sum_{n=0}^{\infty}\frac{2^{2n}B_{2n}}{(2n)!}y^{2n-1} = \frac{1}{y}+\frac{1}{3}y-\frac{1}{45}y^3+\dots, \quad (|y|<\pi) \tag{5.5}$$

which will give rise to an expansion of the form

$$E_T = E_0 + T^2E_2 + T^4E_4 + \dots. \tag{5.6}$$

The zero temperature result was worked out, by inspection, in Ref. [12]:

$$E_0 = -\frac{\lambda_1\lambda_2 ab}{16\pi R}\ln\frac{1-(a-b)^2/R^2}{1+(a+b)^2/R^2}. \tag{5.7}$$

The T^2 term is trivial because it is evaluated by Newton's theorem that a Coulomb potential exterior to a spherically symmetric charge distribution is as though the charge were concentrated at the center:

$$E_2 = -\frac{\lambda_1\lambda_2\pi}{3}\frac{a^2b^2}{R}. \tag{5.8}$$

The T^{2n} term, $n>1$ however, is slightly nontrivial:

$$E_{2n} = -\frac{\lambda_1\lambda_2}{64\pi^3}a^2b^2\frac{(4\pi)^{2n}B_{2n}}{(2n)!}\int d\Omega\, d\Omega'|\mathbf{r}-\mathbf{r}'|^{2n-3}. \tag{5.9}$$

We may evaluate the integrals by expanding in powers of $\hat{a}=a/R$ and $\hat{b}=b/R$:

$$\int d\Omega\, d\Omega'|\mathbf{r}-\mathbf{r}'| = (4\pi)^2R\left[1+\frac{1}{3}(\hat{a}^2+\hat{b}^2)\right] \tag{5.10a}$$

$$\int d\Omega\, d\Omega'|\mathbf{r}-\mathbf{r}'|^3 = (4\pi)^2R^3\left[1+2(\hat{a}^2+\hat{b}^2)+\frac{1}{5}\hat{a}^4+\frac{2}{3}\hat{a}^2\hat{b}^2+\frac{1}{5}\hat{b}^4\right] \tag{5.10b}$$

$$\begin{aligned}\int d\Omega\, d\Omega'|\mathbf{r}-\mathbf{r}'|^5 = (4\pi)^2R^5\Big[&1+5(\hat{a}^2+\hat{b}^2)+3\hat{a}^4+10\hat{a}^2\hat{b}^2+3\hat{b}^4\\ &+\frac{1}{7}\hat{a}^6+\hat{a}^2\hat{b}^2(\hat{a}^2+\hat{b}^2)+\frac{1}{7}\hat{b}^6\Big],\end{aligned} \tag{5.10c}$$

and so on. The reason these are polynomials is evident when one considers the multipole expansion of the Coulomb potential—See, for example, Chap. 22 of Ref. [25]. For general formulas for such moments, see the Appendix.

By computing further terms in the sequence of polynomials, we are able to recognize the pattern:

$$\frac{1}{(4\pi)^2R^{2n+1}}\int d\Omega\, d\Omega'|\mathbf{r}-\mathbf{r}'|^{2n+1} = \sum_{p=0}^{n+1}\sum_{q=0}^{p}A(n,p,q)\hat{a}^{2(p-q)}\hat{b}^{2q}, \tag{5.11}$$

where

$$A(n,p,q) = \frac{(2n+2)!}{(2n-2p+2)!(2p-2q+1)!(2q+1)!}. \tag{5.12}$$

When this is inserted into the low temperature expansion, we can remarkably sum the series:

$$\begin{aligned}E_T = -\frac{\lambda_1\lambda_2}{16\pi}\frac{ab}{R}\Big\{&\ln\frac{1-(a-b)^2/R^2}{1-(a+b)^2/R^2}\\ &+f(2\pi T(R+a+b))+f(2\pi T(R-a-b))-f(2\pi T(R-a+b))-f(2\pi T(R+a-b))\Big\},\end{aligned} \tag{5.13}$$

where f is

$$f(y)=\sum_{n=1}^\infty \frac{2^{2n}B_{2n}}{2n(2n-1)(2n)!}y^{2n}, \tag{5.14}$$

which is obtained from the second antiderivative of the hyperbolic cotangent:

$$y\frac{d^2}{dy^2}f(y)=\coth y-\frac{1}{y},\quad f(0)=f'(0)=0. \tag{5.15}$$

Although the power series expansion (5.14) is valid only for sufficiently low temperatures $2T(R+a+b)<1$, the solution of the differential equation is valid for all values of T.

For sufficiently high temperatures we can replace the hyperbolic cotangent in the differential equation by 1, and then

$$f(y)\sim y\ln y+\ln y+Ay+B,\quad y\gg 1, \tag{5.16}$$

where A and B are integration constants that do not contribute to Eq. (5.13). When this asymptotic solution is inserted into Eq. (5.13) the zero temperature logarithm cancels out, and we are left with

$$E_T\sim-\frac{\lambda_1\lambda_2 ab}{8}T\left[\ln\frac{R^2-(a+b)^2}{R^2-(a-b)^2}+\frac{a}{R}\ln\frac{(R+b)^2-a^2}{(R-b)^2-a^2}+\frac{b}{R}\ln\frac{(R+a)^2-b^2}{(R-a)^2-b^2}\right],\quad T\to\infty. \tag{5.17}$$

This result may be derived directly from the high-temperature form

$$E_T\sim-\frac{\lambda_1\lambda_2 Ta^2b^2}{32\pi^2}\int d\Omega\, d\Omega'\frac{1}{|\mathbf{r}-\mathbf{r}'|^2},\quad T\to\infty. \tag{5.18}$$

This again may be worked out by expanding in powers of the radii of the spheres. Computing the first several terms reveals the pattern:

$$\int d\Omega\, d\Omega'\frac{1}{|\mathbf{r}-\mathbf{r}'|^2}=\frac{(4\pi)^2}{R^2}\sum_{n=0}^\infty\frac{1}{(n+1)(2n+1)}\sum_{m=0}^n\frac{1}{2}\binom{2n+2}{2m+1}\hat{a}^{2(n-m)}\hat{b}^{2m}. \tag{5.19}$$

This sum is almost identical to that found for spheres at zero temperature, as seen in Eq. (6.15) of Ref. [12], which led to Eq. (5.7), except for the appearance of $1/(2n+1)$ here. Therefore, the former series must be obtained from the present series by differentiation. Denoting the double sum in Eq. (5.19) by S, it must be true that

$$R^2\frac{\partial}{\partial R}\frac{S}{R}=\frac{R^2}{4ab}\ln\left(\frac{1-(a+b)^2/R^2}{1-(a-b)^2/R^2}\right), \tag{5.20}$$

where S is $R^2/4ab$ times the square-bracketed quantity in Eq. (5.17). This equality is, in fact, easily verified. See the Appendix for the generalization of this result.

We compare the general form [obtained by numerically integrating Eq. (5.15)] and the high-temperature limiting form (5.17) in Fig. 1.

VI. CONCLUSIONS

We have shown that exact results may be found in weak coupling for the quantum vacuum forces between nontrivial bodies not only at zero temperature, but at finite temperature. We have shown that the exact form of the proximity force approximation holds exactly for all temperatures for the force between an infinite plane surface and an arbitrarily curved one. We have also computed the force between two semitransparent spheres at arbitrary temperatures, and obtain remarkably simple, closed-form expressions. The PFA equivalence evidently will hold for tenuous dielectric bodies in electromagnetism, and closed-form finite temperature results may be easily obtained between dielectric bodies as well.

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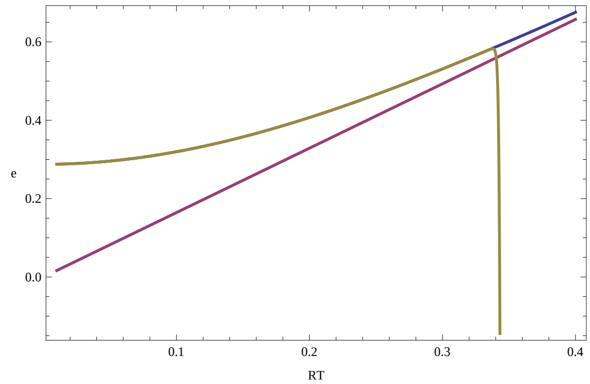

FIG. 1: Comparison between the general and high temperature forms of the energy, as a function of RT. Energies are shown for $a = b = R/4$. The high temperature result is linear in T. Also shown is the power series expansion [Eq. (5.14) truncated at 200 terms], which diverges in this case at $RT = 1/3$. Plotted is $e = -16\pi RE/(\lambda_1\lambda_2 a^2)$.

Appendix A: Mean powers of distances between points on spheres

In Sec. V we used exact evaluations of mean distances, defined by

$$\int d\Omega\, d\Omega' |\mathbf{r} - \mathbf{r}'|^p = (4\pi)^2 R^p P_p(\hat{a}, \hat{b}), \tag{A1}$$

where R is the distance between the centers of the two nonoverlapping spheres, of radii a and b, respectively. Here $\hat{a} = a/R$ and $\hat{b} = b/R$, and $P_p(\hat{a}, \hat{b})$ can in general be represented by the infinite series

$$P_p(\hat{a}, \hat{b}) = \sum_{n=0}^{\infty} \frac{2}{(2n+2)!} \frac{\Gamma(2n-p-1)}{\Gamma(-p-1)} Q_n(\hat{a}, \hat{b}). \tag{A2}$$

Here the homogeneous polynomials Q_n are

$$Q_0 = 1, \tag{A3a}$$
$$Q_1 = 2(\hat{a}^2 + \hat{b}^2), \tag{A3b}$$
$$Q_2 = 3\hat{a}^4 + 10\hat{a}^2\hat{b}^2 + 3\hat{b}^4, \tag{A3c}$$
$$Q_3 = 4\hat{a}^6 + 28\hat{a}^4 b^2 + 28\hat{a}^2\hat{b}^4 + 4\hat{b}^6, \tag{A3d}$$

or in general,

$$Q_n = \frac{1}{2} \sum_{m=0}^{n} \binom{2n+2}{2m+1} \hat{a}^{2(n-m)} \hat{b}^{2m}. \tag{A4}$$

We can easily see the following recursion relation holds:

$$P_{p-1}(\hat{a}, \hat{b}) = \frac{R^{-p}}{1+p} \frac{\partial}{\partial R} R^{1+p} P_p(\hat{a}, \hat{b}), \tag{A5}$$

since Q_n is homogeneous in R of degree $-2n$.

For p a non-negative integer, P_p is a polynomial of degree $2\lceil p/2 \rceil$, and we can immediately find

$$P_p(\hat{a}, \hat{b}) = \frac{1}{4\hat{a}\hat{b}} \frac{1}{(p+2)(p+3)} \left[(1+\hat{a}+\hat{b})^{p+3} + (1-\hat{a}-\hat{b})^{p+3} - (1-\hat{a}+\hat{b})^{p+3} - (1+\hat{a}-\hat{b})^{p+3} \right], \quad p = 0, 1, 2, \dots. \tag{A6}$$

For p a negative integer, we have

$$P_{-1} = 1, \tag{A7a}$$

$$P_{-2} = \frac{1}{4\hat{a}\hat{b}}\left[\ln\frac{1-(\hat{a}+\hat{b})^2}{1-(\hat{a}-\hat{b})^2} + \hat{a}\ln\frac{(1+\hat{b})^2-\hat{a}^2}{(1-\hat{b})^2-\hat{a}^2} + \hat{b}\ln\frac{(1+\hat{a})^2-\hat{b}^2}{(1-\hat{a})^2-\hat{b}^2}\right], \tag{A7b}$$

$$P_{-3} = -\frac{1}{4\hat{a}\hat{b}}\ln\frac{1-(\hat{a}+\hat{b})^2}{1-(\hat{a}-\hat{b})^2}, \tag{A7c}$$

$$P_{-4} = \frac{1}{[1-(\hat{a}+\hat{b})^2][1-(\hat{a}-\hat{b})^2]}, \tag{A7d}$$

and further expressions can be obtained by use of Eq. (A5).

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